Convex Synthesis of SNI Controllers Based on Frequency-Domain Data: MEMS Nanopositioner Example

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Abstract—We report a procedure to design fixed-structure strictly negative imaginary (SNI) controllers for collocated, highly resonant systems based on frequency response data. We formulate the problem in a two-input two-output (TITO) framework that includes robustness and stability constraints. The controller synthesis is cast as a convex optimization problem using convex approximation techniques. The measured frequency response of a two-degree-of-freedom (2-DOF) microelectromechanical system (MEMS) nanopositioner is used to compute the frequency response of the controller by minimizing the difference between the actual and desired closed-loop responses in an $H_2$ sense. By including the negative imaginary (NI) stability criterion as a constraint in the optimization algorithm, closed-loop stability is guaranteed. Performance of the synthesized controllers is verified in time and frequency domains through closed-loop experiments with the MEMS nanopositioner.

Index Terms—Convex optimization, damping controller, data-driven control, microelectromechanical system (MEMS), negative imaginary (NI) systems.

I. INTRODUCTION

DATA-DRIVEN control design techniques have emerged in recent years to address drawbacks associated with model-based control design methods and prevent performance degradation in the presence of high-order system dynamics, environmental changes, and plant uncertainties and nonlinearities [1]. These techniques rely on the measured input–output data to circumvent the requirement for reidentification of the system and redesigning of the controller. They offer an alternative approach to developing novel control design techniques for practical applications where low-cost, easy-to-install, and ready-to-use controllers are needed.

A data-driven control design approach based on the frequency response data and convex optimization is reported in [2]–[5]. The control design problem is formulated as a convex–concave optimization problem and convexified by linearizing the concave component around an initial controller. This approximation method is introduced in [2] to linearize nonconvex matrix inequalities for multiple-input multiple-output (MIMO) PID control design. The optimization procedure converges to a local optimum of the original nonconvex problem in an iterative manner. Although there is no guarantee for a globally optimal solution, the resulting controller is promising in practice. In [3], this data-driven control design approach is extended to fully parameterized controllers, not limited to PID or linearly parameterized (LP) controllers. Different performance criteria can be enforced with this data-driven control design procedure, and the framework can incorporate robust control design for plants with multimodal uncertainty. Performance of the synthesized controllers is validated experimentally with an atomic force microscope (AFM) piezoelectric tube scanner [6], [7], and a two-degree-of-freedom (2-DOF) gyroscope [8].

In data-driven control design methodologies, in general, there is no guarantee that the resulting controller will stabilize the plant, and the stability of the controlled system is typically investigated through rigorous mathematical analysis [1]. For instance, in [3], closed-loop system stability is analyzed based on the generalized Nyquist stability criterion and incorporated as a linearized constraint in the optimization algorithm. This issue can be addressed using negative imaginary (NI) system theory [9] for a class of flexible structures with collocated and compatible actuator–sensor pairs, e.g., position sensors and force actuators. A linear time-invariant (LTI) system is considered to be NI if its imaginary part is negative for all frequencies between 0 and $\infty$. The NI stability theorem states that a positive feedback interconnection of an NI plant with a strictly negative imaginary (SNI) controller is internally stable, provided that the dc loop gain is less than unity. Stability robustness of interconnected NI systems has been leveraged to synthesize novel model-based and data-driven control structures [10], [11].

Among recent studies on model-based SNI control synthesis, Mabrok [12] employed nonlinear optimization techniques to design fixed-structure single-input single-output (SISO) SNI controllers that minimize the $H_2$ norm of a cost function and satisfy the dc gain condition. The controller is simulated on a flexible robotic arm system. In [13], an optimal SISO SNI controller is designed by defining a convex objective function...
and linear matrix inequality (LMI) constraints to guarantee the SNI properties and the dc gain condition. In this approach, it is assumed that a state-space model of the plant and the $H_\infty$ optimal controller are known in advance. The goal is to minimize the distance between the SNI and the $H_\infty$ optimal controllers. To investigate the performance of the synthesized controller, we simulate positive feedback interconnection of the controller with a multi-joint manipulator. An extension of [13] is presented in [14], where weighted $H_2$ and $H_\infty$ performance measures are considered to design an optimal SNI controller. The proposed synthesis method is applied to vibration control of a simply supported Euler–Bernoulli beam.

Besides model-based techniques, the data-driven control synthesis methodology has been applied to NI systems without explicit knowledge of the plant model and controller structure [15]. Using the frequency response of the plant, a nonconvex optimization problem is convexified by changing the optimization variable and solved at each frequency point to determine the controller frequency response. The nonconvex problem is constrained by the NI condition and a bound on the controller magnitude response. Finally, a modified subspace system identification algorithm, including NI constraints, is used to fit a transfer function to the controller response. The data-driven approach is modified in [16] and extended in [17] by applying the small-gain constraint. To fit a model to the controller frequency response, a prediction error minimization estimation and a standard system identification method are used in [16] and [17], respectively. The drawback with this data-driven control design is the complexity of fitting an accurate and low-order NI model to the controller frequency response using standard system identification techniques.

In this article, a fixed-structure control synthesis procedure is presented for a class of MIMO plants satisfying the NI property. The controllers asymptotically stabilize the system in the presence of input disturbances. The design procedure is based on the measured frequency response of the NI system without the requirement for a parametric model. Inspired by the data-driven approach and approximation technique discussed in [2] and [3], the control synthesis problem is cast as a convex optimization problem using LMI constraints. The resulting controller is a suboptimal solution to the nonconvex problem. The performance criterion is to minimize the difference between the desired and actual closed-loop frequency responses in an $H_2$ sense. The optimization problem enforces the NI internal stability condition and a constraint on the actuator effort. The controllers are experimentally implemented to augment damping to the resonant mode of a 2-DOF microelectromechanical system (MEMS) nanopositioner. Experiments are conducted in time and frequency domains to justify the efficacy of the synthesized controllers.

This article is organized as follows. Section II reviews the existing NI system definition and stability theorem for continuous- and discrete-time systems and presents the notations used in the rest of this article. In Section III, a class of fixed-structure SNI controllers is introduced. Following the problem statement presented in Section IV, the convex formulation of control design is discussed in Section V. In Section VI, the data-driven approach is employed to design the SNI controllers to damp the resonant mode of a 2-DOF MEMS nanopositioner using frequency response data. This section also includes the experimental results and discussion. Finally, Section VII concludes this article.

II. PRELIMINARIES

A. Notations

Here, $G(j\omega)$ refers to the frequency response of a continuous-time rational transfer function $G \in \mathbb{C}^{m \times m}$, where $m$ is the number of inputs and outputs. $G(e^{j\omega})$ is the discrete-time frequency response of $G$. Assuming adequately fast sampling, $G(e^{j\omega})$ and $G(j\omega)$ are nearly identical. We assume that $G$ is stable, bounded at all frequencies, and has no poles on the stability boundary.

The controller is assumed to be a fixed-order and fixed-structure rational transfer function factorized as

$$K = X(z)Y(z)^{-1}$$

where $X, Y \in \mathbb{C}^{m \times m}$ are discrete-time polynomial matrices defined as

$$X(z) = X_0 z^p + \cdots + X_1 z + X_0$$

$$Y(z) = Y_0 z^p + \cdots + Y_1 z + Y_0$$

where $X_j, Y_i \in \mathbb{R}^{m \times m}$ denote the controller parameters.

The infinity norm and two norm of $H(e^{j\omega})$ are defined as

$$\|H\|_2^2 = \frac{1}{2\pi} \int_{\Omega} \text{Tr}[H^*(e^{j\omega})H(e^{j\omega})]d\omega$$

$$\|H\|_{\infty} = \sup_{\omega \in \Omega} \|H(e^{j\omega})\|$$

$$\omega \in \Omega = \left \{ \omega : -\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s} \right \}$$

where $T_s$, $\text{Tr}()$, and $\bar{\sigma}(\cdot)$ denote the sampling time, trace of a matrix, and maximum singular value of a matrix, respectively.

B. Definitions and Background Theorems

Continuous-time NI and SNI systems are defined as follows.

Definition 1 [18]: A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied.

1) $G(s)$ has no poles at the origin and in $\Im(s) > 0$.
2) $|G(j\omega) - G^*(j\omega)| \geq 0$ for all $\omega > 0$ except for those where $j\omega$ is a pole of $G(s)$.
3) If $j\omega_0$, $\omega_0 > 0$, is a pole of $G(s)$, then at most a simple pole and the residue matrix $K_0 = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)G(s)$ is positive semidefinite Hermitian.

Definition 2 [18]: A square transfer function matrix $G(s)$ is called SNI if the following conditions hold.

1) $G(s)$ has no poles in $\Im(s) \geq 0$.
2) $|G(j\omega) - G^*(j\omega)| > 0$ for all $\omega > 0$.

These definitions may be extended to discrete-time systems.

Definition 3 [19], [20]: A discrete-time, real-rational, proper transfer function, $G(z)$, is discrete-time NI (D-NI) if the following conditions hold.

1) $G(z)$ has no poles in $|z| > 1$.
2) $|G(e^{j\omega}) - G^*(e^{j\omega})| > 0$ for all $\omega \in (0, \pi)$ except for the values of $\omega_0$, where $z = e^{j\omega_0}$ is a pole of $G(z)$.
3) If \( z_0 = e^{i\omega_0} \) with \( \omega_0 \in (0, \pi) \) is a pole of \( G(z) \), then it is a simple pole and the residue matrix
\[
[\bar{G}] = z_0^{-1} \lim_{z \to z_0} (z - z_0)jG(z)
\]
is Hermitian and positive semidefinite.
4) If \( z = 1 \) is a pole of \( G(z) \), then
\[
\lim_{s \to \infty} (s - 1)^2G(s) = \text{Hermitian and positive semidefinite},
\]
and
\[
\lim_{s \to \infty} (z - 1)^2G(z) = 0 \text{ for all integer } k \geq 3.
\]
5) If \( z = -1 \) is a pole of \( G(z) \), then
\[
\lim_{s \to \infty} (s + 1)^2G(s) = \text{Hermitian and negative semidefinite},
\]
and
\[
\lim_{s \to \infty} (z + 1)^2G(z) = 0 \text{ for all integer } k \geq 3.
\]

**Definition 4** [19], [20]: A discrete-time, real-rational, proper transfer function \( G(z) \) is discrete-time SNI (D-SNI) if the following conditions hold.

1. \( G(z) \) has no poles in \(|z| \geq 1\).
2. \( j[G(e^{i\omega}) - G^*(e^{i\omega})] > 0 \) for all \( \omega \in (0, \pi) \).

The following lemma describes the one-to-one mapping between continuous- and discrete-time NI transfer functions [21].

**Lemma 1**: A continuous-time NI transfer function matrix \( G(s) \) transforms into a discrete-time NI transfer function \( G(z) \) using the bilinear transformation \( s = \frac{2z}{1+z} \). Conversely, an NI transfer function \( G(z) \) transforms into a continuous-time NI transfer function matrix \( G(s) \) by the bilinear transformation \( z = \frac{1+is}{1+is} \).

Suppose that an NI system \( G \) is in a positive feedback loop with an SNI system \( K \), as shown in Fig. 1. The following theorem establishes the conditions under which the internal stability of the closed-loop system is guaranteed.

**Theorem 2** [9]: Given a continuous-time NI system \( G(s) \) and an SNI system \( K(s) \) that also satisfy \( G(\infty)K(\infty) = 0 \) and \( K(\infty) > 0 \)
\[
(G(s), K(s)) \text{ is internally stable } \Leftrightarrow \lambda_i(G(0)K(0)) < 1. \tag{6}
\]
where \( \lambda_i(.) \) denotes the largest eigenvalue of a matrix with all real eigenvalues.

**Proof**: See [9, Th. 5]. \( \Box \)

The results may be extended to discrete-time systems.

**Theorem 3** [20]: Suppose that \( G(z) \) is a discrete-time, real-rational, proper NI system without poles at \( z = \pm 1 \); also, suppose that \( K(z) \) is a discrete-time, real-rational, proper SNI system. Then, the positive feedback interconnection of \( G(z) \) and \( K(z) \) is internally stable if and only if

\[
I - G(-1)K(-1) \text{ is nonsingular}
\]
\[
\lambda_i[(G(1)K(-1) - I)(I - G(-1)K(-1)^{-1})] < 0, \text{ and}
\]
\[
\lambda_i[(K(1)G(1) - I)(I - K(-1)G(1)^{-1})] < 0
\]

**Proof**: Refer to [20, Th. 6]. \( \Box \)

These stability results can be simplified under some assumptions as described in the following.

**Corollary 3.1** [20]: Let \( G(z) \) be a discrete-time, real-rational, proper NI system without poles at \( z = \pm 1 \); also suppose that \( K(z) \) is a discrete-time, real-rational, proper SNI system. Also, assume that \( G(-1)K(-1) = 0 \) and \( G(1) > 0 \). Then, \([G(z), K(z)] \) is internally stable if and only if \( \lambda_i[(G(1)K(1))] < 1 \).

**Proof**: Refer to [20, Corollary 7]. \( \Box \)

In Section V, we will use a part of the proof of Corollary 3.1 to derive a stability constraint for the control design problem studied here.

### III. Negative Imaginary Controllers

Transfer functions of highly resonant systems with compatible and collocated actuator–sensor pairs typically satisfy the NI conditions. It is possible to construct SNI controllers in positive feedback that guarantees the internal stability of the closed-loop system and lower quality factor of structural modes as required. Although resonance frequencies and dynamics of the underlying system could change depending on the operating conditions, collocation of actuators and sensors preserves the NI property of the system. Therefore, closed-loop stability will be preserved, whereas the closed-loop performance may deteriorate. This property makes the SNI controllers highly desirable in vibration control applications.

In this section, we introduce two such controllers, a positive position feedback controller (PPF) and a specific form of phase-lead controller (PLC). The PPF controller has been extensively used in vibration control applications [22]–[26]. The PLC is a first-order SNI controller that can increase damping and reduce the system sensitivity to flicker noise in a manner similar to resonant controllers [27]. These parameterized controllers are low-order and relatively straightforward to implement.

#### A. PPF Controller

The PPF controller is a popular vibration control method originally proposed in [28]. Its sharp beyond-the-bandwidth roll-off provides a high level of performance by mitigating the risk of closed-loop instabilities arising from unmodeled high-frequency dynamics [23]. This controller has been employed in a wide range of applications, e.g., vibration control of a single-link flexible manipulator [22], active vibration control of a grid structure equipped with piezoceramic sensors and actuators [29], active damping of resonant modes, and compensation for cross coupling in a piezoelectric tube scanner in a commercial AFM [24], and active damping of a cantilever beam for tapping mode atomic force microscopy [26]. The NI property of this control scheme is studied in detail in [10]. A continuous-time multivariable PPF controller is described as [10], [23]

\[
K_{PPF} = \sum_{i=1}^{S} \frac{\psi_i \psi_i^T}{s^2 + d_i s + \omega_i^2} \tag{7}
\]

\[
= \Psi[s^2 I + Ds + \Omega]^{-1} \Psi^T \tag{8}
\]
where \( \tilde{N} \) is the number of resonant modes, \( m \) is the number of actuators and sensors, \( \psi_i \in \mathbb{R}^{m \times 1}, \quad \Psi = [\psi_1, \psi_2, \ldots, \psi_N] \), and \( d_i, \omega_i, D, \Omega > 0 \).

### B. Phase-Lead Controller

This controller has a multivariable form of

\[
K_{\text{PLC}} = -s[I + B]^{-1}A
\]

where \( A, B > 0 \) and \( A, B \in \mathbb{R}^{m \times m} \).

In [30] and [31], a SISO version of this controller is designed and implemented to augment the damping of the fundamental resonant mode of a MEMS nanopositioner.

### IV. PROBLEM STATEMENT

Considering an NI LTI two-input two-output (TITO) strictly proper system with a highly resonant frequency response \( G(e^{j\omega}) \), \( \omega \in \Omega \), the goal is to synthesize the fixed-structure SNI controllers, introduced in Section III, with frequency response of \( K(e^{j\omega}) = X(e^{j\omega})Y^{-1}(e^{j\omega}) \) satisfying the following specifications.

(i) The main objective is to suppress the resonant mode by shaping the closed-loop response so that the distance between the desired and actual closed-loop frequency responses is minimized in an \( H_2 \) sense (4), i.e.,

\[
\min_{X, Y} \| T - T_d \|_2^2
\]

where \( T \) and \( T_d \) are the actual and desired closed-loop frequency response, respectively. The desired closed-loop transfer function is selected to achieve the requisite bandwidth and decoupling.

(ii) To avoid actuator saturation and subsequent closed-loop instabilities arising from a sudden change in the level of input disturbance, the following constraint is imposed to limit actuator effort:

\[
\| WU \|_{\infty} < 1
\]

where \( U \) is the load disturbance sensitivity function and \( W \) is a weighting filter that can be a scalar or an arbitrary transfer function [3].

(iii) To ensure internal stability of the feedback control system, the following NI stability condition must be satisfied:

\[
\tilde{\lambda}[G(1)K(1)] < 1.
\]

Since the plant is assumed to be NI and the controller is SNI, by construction, there is no need to set an additional constraint to enforce the SNI property of the controller. A synthesized controller satisfying conditions (i)–(iii) above will guarantee the desired level of performance as well as stability and robustness of the closed-loop system. We formulate the controller synthesis problem as a convex optimization problem based on LMIs, which can be solved efficiently using standard solvers.

### V. CONTROL DESIGN ALGORITHM

We employ the data-driven control design methodology described in [3] to synthesize the PPF and PLC controllers for TITO highly resonant NI systems. Since the controller implementation will be in discrete time, we first discretize the plant to allow for discrete-time controller synthesis. Lemma 1 implies that, under bilinear transformation, the NI property carries over from continuous time to discrete time. Therefore, the resulting system is NI, setting the scene for synthesis of a discrete-time SNI controller.

Here, the system to be controlled has one fundamental resonant mode; thus, \( K_{\text{PPF}} \) in (7) can be written as

\[
K_{\text{PPF}} = \frac{\psi_1 \psi_1^T}{s^2 + d_1s + \omega_1^2}.
\]

For multiresonant systems, the number of PPF controllers is determined based on the number of modes to be controlled, and the equivalent PPF controller is obtained by parallel connection of individual controllers designed for each mode. Since the closed-loop system resulting from the positive interconnection of an NI plant with an SNI controller is also NI [10], the proposed methodology can be extended to design the equivalent multivariable PPF controller.

As described in Appendix A, by applying the bilinear transformation to (13) and factorizing the controller as \( K = XY^{-1} \), the corresponding PPF polynomial matrices can be represented by

\[
K_{\text{PPF}} = X_{\text{PPF}}Y_{\text{PPF}}^{-1}
\]

where \( X_{\text{PPF}} = X_2z^2 + X_1z + X_0, \quad X_2 = X_0, \quad X_1 = 2X_0, \quad X_0 > 0 \)

\[
Y_{\text{PPF}} = Y_2z^2 + Y_1z + Y_0, \quad Y_0, Y_2 > 0
\]

where \( X_i \in \mathbb{R}^{2 \times 2} \) is symmetric and \( Y_i \in \mathbb{R}^{2 \times 2} \) is a diagonal matrix with similar entries. Similarly, the corresponding PLC polynomial matrices are obtained using the bilinear transformation and assuming that \( A \) and \( B \) are square matrices and \( B \) is diagonal

\[
K_{\text{PLC}} = X_{\text{PLC}}Y_{\text{PLC}}^{-1}
\]

where \( X_{\text{PLC}} = X_1z + X_0, \quad X_1 = -X_0, \quad X_0 > 0 \)

\[
Y_{\text{PLC}} = Y_1z + Y_0, \quad Y_1 > 0
\]

Note that although the procedure followed in this article considers a TITO system, the methodology can be equally extended to enable SNI controller synthesis for square NI systems of arbitrary dimensions.

### A. Convex Formulation of SNI Control Design

We formulate the SNI control synthesis problem as a standard convex optimization problem with LMI constraints as described in [2] and [3]. To achieve the main goal of active damping and disturbance rejection, PPF and PLC controllers are positioned in the feedback loop. This arrangement also allows for a secondary tracking loop, including a high gain or integral action [24], [32]. In the following, we establish how...
the feedback control design problem can be formulated as a convex optimization problem. For simplicity, we discard $e^{i\omega}$ in the following formulations.

To cast the first objective in Section IV as an LMI constraint, we can rewrite (10) as

$$\min_{\omega} \int_{\omega} \text{Tr}(\Gamma(\omega))d\omega$$

subject to: $[T - T_d]^*[T - T_d] < \Gamma(\omega)$

where $\Gamma(\omega) \in \mathbb{R}^{2\times2} > 0$ is an unknown matrix function and $T = G(I - GK)^{-1}$ is the closed-loop transfer function. Substituting $T$ and $K = XY^{-1}$ in (20), we have

$$[GY(Y - GX)^{-1} - T_d]^*[GY(Y - GX)^{-1} - T_d] < \Gamma(\omega)$$

As described in Appendix B, the above constraint can be written as the following LMI:

$$\begin{bmatrix} \Gamma(\omega) \\ (GY - T_d Z)^* & Z^* Z_c + Z_e^* Z - Z_e^* Z_c \end{bmatrix} > 0 \quad \forall \omega \in \Omega$$

where $Z = Y - GX$, $Z_c = Y - GX_e$, and $K_c = X_e Y^{-1}$ is the initial controller [3].

The next criterion is to limit the maximum level of actualization. To enforce this, we may write constraint (11) as

$$[WU]^*[WU] < I.$$ (23)

Substituting $U = (I - GK)^{-1}$ and $K = XY^{-1}$ in (23) and using convex–concave approximation around $Z_c$ [6], we can write (23) as a convex LMI constraint

$$\begin{bmatrix} Z^* Z_c + Z_e^* Z - Z_e^* Z_c \\ (WY)^* \end{bmatrix} > 0.$$ (24)

To enforce the NI stability condition (12) as a linear constraint, we use the following fact (see [20, Proof of Corollary 3.1]):

$$\bar{\lambda}[G(1)K(1)] < 1 \iff K(1) - G(1)^{-1} < 0.$$ (25)

Substituting $K = XY^{-1}$ and assuming $\det(Y) \neq 0$, $\forall \omega \in \Omega$, we can express the right-hand side of (25) as

$$X(1) - G(1)^{-1}Y(1) < 0$$

which is affine in $X$ and $Y$. Accordingly, all the control objectives are formulated as convex LMI constraints that can be solved efficiently using standard solvers.

This optimization problem is a semi-infinite problem in the sense that it consists of an infinite number of constraints [2], [3]. In order to solve this problem, the frequency range is sampled logarithmically to generate a finite but relatively large set of $N$ frequency points allowing the semi-infinite constraints to be replaced with a finite set of constraints at each frequency point. Applying this procedure to PPF controller synthesis problem results in

$$\min_{X,Y} \sum_{k=1}^{N} \text{Tr}(\Gamma_k)$$

subject to: $X_2 = X_0$, $X_1 = 2X_0$, $X_0 > 0$

$$Y_0, Y_2 > 0$$

$$X(1) - G(1)^{-1}Y(1) < 0$$

$$\begin{bmatrix} \Gamma_k & (GY - T_d Z) \\ (GY - T_d Z)^* & Z^* Z_c + Z_e^* Z - Z_e^* Z_c \end{bmatrix}(\omega_k) > 0$$

$$\begin{bmatrix} Z^* Z_c + Z_e^* Z - Z_e^* Z_c \\ (WY)^* \end{bmatrix}(\omega_k) > 0$$

$k = 1, 2, \ldots, N$ (27)

where $\Gamma_k$ denotes the sampled $\Gamma(\omega)$ at $\omega_k$, with $\{\omega_k, k = 1, 2, \ldots, N\}$ the frequency points in the sampled optimization problem. The PLC convex synthesis problem is similar to that of PPF and can be obtained by replacing the first two constraints in (27) with

$$X_1 = -X_0, X_0 > 0$$

$$Y_1 > 0.$$ (28)

There are several points that require attention. The internal stability of the closed-loop system is guaranteed due to the positive feedback connection of an NI system with an SNI controller. Although the SNI controllers provide robustness against certain uncertainties in system dynamics, e.g., variations in resonance frequencies or damping ratios, the design algorithm can incorporate a set of frequency response functions (FRFs) of the plant at different measurements to enforce specific robustness criteria [3]. The design starts with an initial controller, which must satisfy the constraints. Since the controller structure is fixed, we initialize the optimization with a feasible controller possessing a small gain. The resulting controller is not necessarily the globally optimal solution to the original nonconvex problem since we employ a convex approximation technique. We solve the convex optimization problem iteratively using the resulting controller as the new initial controller for the next iteration. In this way, the iterations are feasible, and the algorithm converges to a solution that meets the desired performance [2].

VI. DATA-DRIVEN DAMPING CONTROL OF A MEMS NANOPositioner

In this section, we employ the data-driven approach to design the PPF and PLC controllers and damp the resonant mode of a 2-DOF MEMS nanopositioner. This improves the bandwidth of the nanopositioner, enabling its use in applications such as high-speed atomic force microscopy [33], [34]. In closed-loop experiments with this MEMS nanopositioner, the performance of the synthesized controllers is investigated in time and frequency domains.

A. MEMS Nanopositioner

We design the damping controllers for a 2-DOF MEMS nanopositioner with collocated sensors and actuators. A scanning electron microscope (SEM) image of the device is
shown in Fig. 2(a). It features a stage with dimensions of 1.8 mm × 1.8 mm at the center and four electrostatic actuators that generate bidirectional motion along the X- and Y-axes with the maximum linear displacement range of 12 μm in each axis. The bilateral actuation mechanism enables us to address the inherent quadratic nonlinearity of electrostatic actuators [35]. In order to measure the in-plane deflection of the stage, on-chip bulk piezoresistive sensors are integrated with the device. The design and characterization of the device are detailed in [36].

B. Frequency Response

In order to obtain the FRF of the MEMS nanopositioner, we consider the device as a TITO system where the inputs are the voltages applied to the actuators through a high-voltage amplifier, and the outputs are the X- and Y-axis sensor output voltages. The FRF of the nanopositioner is determined as

\[
G(jω) = \begin{bmatrix}
G_{xx}(jω) & G_{xy}(jω) \\
G_{yx}(jω) & G_{yy}(jω)
\end{bmatrix} = 
\begin{bmatrix}
V_x(jω) & V_y(jω) \\
U_x(jω) & U_y(jω)
\end{bmatrix}
\begin{bmatrix}
V_x(jω) \\
V_y(jω)
\end{bmatrix}
\]

where \(V_{x,y}\) and \(U_{x,y}\) are Fourier transforms of X- and Y-axis sensor outputs and actuation voltages, respectively. The FRFs are obtained using a fast Fourier transform (FFT) analyzer with single-channel excitation. A wideband sinusoidal chirp signal of varying frequency is applied to the actuators through a high-voltage amplifier with a voltage gain of 20, and the corresponding responses from a piezoresistive readout circuit are recorded over a bandwidth of 25 kHz. Then, the frequency response is sampled at \(N = 2048\) logarithmically spaced frequency points up to 25 kHz with the sampling rate of 50 kHz using the \(spa\) command in MATLAB, which estimates the frequency response at given frequencies. This process reduces noise in capturing cross-coupling FRFs due to the low signal-to-noise ratio. The data are used later in the optimization algorithm.

Note that the FRF of the continuous-time system \(G(jω)\) can be assumed identical to that of discrete-time system \(G(e^{jω})\) since the Nyquist criterion is satisfied. Fig. 3 shows the measured and sampled frequency responses of the nanopositioner. The fundamental resonance frequencies of the nanopositioner are at 3698 and 3645 Hz for X- and Y-axis, respectively. We observe that the frequency response of the nanopositioner in the X-axis is almost similar to the frequency response in the Y-axis. Although the cross couplings between the lateral axes are negligible at low frequencies, they are significant at resonance frequencies and cannot be ignored.

Under ideal assumptions, a flexure-guided nanopositioner with collocated sensors and actuators and lightly damped modes should behave as an NI system. To verify the NI property of the MEMS nanopositioner [37], eigenvalues of the matrix \(j[G(jω) - G^*(jω)]\) are plotted in Fig. 4. From the insets, we note that the system is NI up to 3976 Hz with the phase response being limited to the 0° and −180° range (see Fig. 3). Therefore, the internal stability of the closed-loop system is guaranteed in this region with an SNI controller placed in positive feedback. As the frequency increases beyond 3976 Hz, the phase drops slightly, violating the NI property of the plant, as shown in the inset of Fig. 4. However, the deviation from being NI is negligible, offering a long range of stability for the controller parameters [30]. In the control synthesis algorithm presented here, constraints are also imposed to limit the actuator effort (11) and shape the closed-loop frequency response (10), preventing the closed-loop system from becoming unstable in those regions where the plant violates the NI property. Moreover, according to the stability criterion proposed in [3], stability is guaranteed here since \(\det(Y) \neq 0\) and the order of \(\det(Y)\) is equal to the order of \(\det(Y)\) for both PPF and PLC controllers.

C. Controller Synthesis

The objective is to design multivariable SNI controllers that add significant damping to the system and achieve adequate decoupling between the lateral axes of the MEMS nanopositioner. This is done by shaping the closed-loop frequency response so that the \(H_2\) norm of the difference between the actual and the desired transfer functions, i.e., \(\|T - T_d\|_2^2\), is minimized. For PLC synthesis, the desired transfer function...
Fig. 3. Frequency response of the MEMS nanopositioner up to 25 kHz.

Fig. 4. Eigenvalues of $j(G(j\omega) - G^*(j\omega))$.

is chosen as

$$T_d = \frac{\omega_c}{f\omega + \omega_c}I$$

(30)
to attain the desired closed-loop bandwidth of $f_c = 3700$ Hz. For the PPF controller design, $3.5T_d$ is selected to achieve perfect damping. To ensure the stability of the controlled nanopositioner and prevent saturation under nominal disturbances, a constraint on the actuation voltage level is imposed, as stated in (23). Based on the physical limitations of the nanopositioner’s actuators, the weighting function $W$ is selected as

$$W = [\sigma_{\text{min}}(G(j\omega))/3]I$$

(31)

where $\sigma_{\text{min}}(G(j\omega))$ denotes the minimum singular value of $G(j\omega)$.

We first initialize the continuous-time controllers to satisfy the NI property as well as the other constraints. Then, we obtain the initial polynomial matrices $X_c$ and $Y_c$ in discrete time for the PPF (14) and PLC (17) controllers using the bilinear transformation. Subsequently, the optimization algorithm is solved iteratively using a feasible initial controller. As described in Appendix B, the original nonconvex problem is convexified using the Taylor expansion around the initial controller, and the synthesized controller is a local optimum or saddle point of the original nonconvex optimization problem [3]. Theoretically, any initial controller that satisfies the optimization constraints yields a feasible solution using the iterative method. However, particular initial controllers may result in infeasible solutions due to numerical problems [3]. In some cases, it is possible to relax the constraints to make the solution feasible. Here, we choose the initial controllers that satisfy both the NI property and the constraints. According to (13), the PPF controller is initialized by setting

$$\omega_{10} = \omega_c, \quad d_{10} = 0.2\omega_c.$$  

(32)

To ensure that $\lambda[G(j\omega)K_c(j\omega)] < 1$ is satisfied, the initial $\psi_{10}^T\psi_{10}$ is considered as $\varepsilon\Psi_0$, where $\Psi_0$ is determined by solving a feasibility problem, including the dc gain constraint (6). By setting $\varepsilon = 0.01$, we have

$$\psi_{10}^T\psi_{10} = 0.01\omega_c^2 \times \begin{bmatrix} 1.0277 & 0.0023 \ 0.0023 & 0.9441 \end{bmatrix}.$$  

(33)
The phase-lead damping controller is initialized by selecting
\[
A_0 = \begin{bmatrix} 0.01 & 0.0001 \\ 0.0001 & 0.01 \end{bmatrix}, \quad B_0 = \omega_c I.
\] (34)

Due to its zero at the origin, the dc gain condition is already satisfied. To solve the optimization problems (27) and (28), \(N = 2048\) frequency points are logarithmically sampled in the interval \([12, 25 \times 10^3]\) Hz. The optimization problem is formulated in MATLAB using the YALMIP toolbox [38] and solved with the MOSEK [39], which is capable of solving both linear and quadratic programs with LMIs.

To investigate the effect of the initial parameters on the optimization process, we randomly select initial values that yield feasible initial controllers. Tables I and II represent the initial and final parameters, along with the total number of iterations for PPF and PLC synthesis, respectively. The last row in the tables indicates the selected initial and final parameters reported here. The optimization algorithm converges with all initial values, and the parameters of the synthesized controllers vary slightly at convergence. For a fair comparison, we set a metric to measure the convergence; however, it is possible to stop the optimization algorithm earlier once the desired performance is achieved.

The frequency responses of the PLC and PPF controllers, resulting from the optimization algorithm, are shown in Fig. 5. The synthesis algorithm also returns parameters of the discretized controllers, which enables their immediate use in standard rapid prototyping systems such as dSPACE. For analog implementation of the controllers and frequency response measurement in closed loop, we also determine the continuous-time damping controller parameters. Fig. 5 shows the Bode plots of the continuous-time PPF and PLC controllers based on the frequency responses obtained from the data-driven approach. The damping controllers in (7) and (13) are obtained as
\[
K_{PPF} = \frac{1}{s^2 + 5.1164 \times 10^4 s + 1.016 \times 10^9} \times \begin{bmatrix} 6.931 \times 10^8 \\ 7.609 \times 10^5 \\ 6.246 \times 10^8 \end{bmatrix}, \quad K_{PLC} = -\frac{s}{s + 6.048 \times 10^5} \times \begin{bmatrix} 5.384 \\ -8.764 \times 10^{-5} \\ 5.420 \end{bmatrix}.
\] (35) (36)

D. Experimental Results

In this section, we report the experimentally obtained frequency and step responses of the MEMS nanopositioner in
open loop and with the synthesized controllers. Although the controllers can be directly implemented in dSPACE, measuring the closed-loop FRF using an FFT analyzer resulted in a noticeable lag in the phase response of the system due to the limited sampling rate of dSPACE. In order to obtain accurate closed-loop frequency responses, we implemented the controllers using an Anadigm QuadApex development board containing four field-programmable analog array (FPAA) devices. FPAAAs are reconfigurable switched capacitor devices containing analog blocks that offer field programmability by the interconnection of these blocks. They enable analog controller implementation with a maximum clock rate of 16 MHz. This implementation significantly enhances the control bandwidth in comparison to dSPACE systems.

Fig. 2(b) shows the experimental setup comprising an FPAA development board and a printed circuit board (PCB) that contains actuation and sensing signal paths of the MEMS nanopositioner. Fig. 6 shows a TITO implementation of the closed-loop system with the damping controllers using three FPAA units in series to capture the frequency response of the nanopositioner when the X-axis actuator is excited. Here, $S_x$, $S_y$, $U_x$, and $U_y$ refer to the piezoresistive sensor outputs

Fig. 7. Frequency response of the MEMS nanopositioner in open loop and closed loop with (a) PPF controller and (b) with PLC. The blue solid line (---) represents the measured open-loop FRF and the red dashed line (—) represents closed-loop FRF obtained by implementing the synthesized controllers and the black dashed dotted line (-----) represents the desired magnitude response.
A convolution optimization-based control design algorithm was presented that utilizes the frequency response data obtained from a highly resonant NI system. The method enables us to synthesize a class of SNI controllers to augment significant damping to the lightly damped modes of the system. Positive position feedback connection of the SNI controller with the NI plant guarantees internal stability, and the algorithm selects controller parameters so that the requisite levels of damping are achieved in closed loop. The algorithm allows for constraints on the closed-loop frequency response, maximum actuation level, and NI stability criterion to be cast as a convex optimization problem with LMI constraints. The method was successfully employed to design high-performance positive position feedback and phase-lead controllers for a lightly damped MEMS nanopositioner. In conclusion, we reported a ready-to-use method for the design and direct implementation of SNI controllers based on the measured frequency response data obtained from an NI plant. The controllers can be applied to SISO and square MIMO plants.

APPENDIX A

To extract the numerator and denominator polynomials in (14) and (17), we apply the bilinear transformation to the transfer functions in (9) and (13) and factorize them to $K = XY^{-1}$. Accordingly, by substituting $s = \frac{z-1}{T z+1}$ in (9), we have

$$K_{PLC} = -s[s I + B]^{-1}A = -\frac{2 z-1}{T z+1}\left[\frac{2 z-1}{z}I + B\right]^{-1}A$$

where $T$ is the sampling period. Since $A$ and $B$ are square matrices and $B$ is diagonal, it can be deduced from (37) that

$$X = (-2A)z + 2A, \quad Y = (B + 2I)z + (TB - 2I).$$

Having $A, B > 0$, it is clear that $X_1 = -X_0, X_0 > 0$, and $Y_1 > 0$.

To apply the bilinear transformation to $K_{PPF}$, we consider the case of MIMO plants with single resonant mode, which is investigated in this article. According to (13), we have

$$K_{PPF} = \frac{\psi_1 \psi_1^T}{s^2 + d_1 s + \omega_1^2} = \frac{\psi_1 \psi_1^T}{T^2(z + 1)\psi_1 \psi_1^T}$$

Here, the denominator is a scalar polynomial. For simplicity, we consider $Y$ as a diagonal matrix with the entries equal to the denominator in (39). This helps us to formulate the controller as $K_{PPF} = XY^{-1}$ without changing the integrity.

VII. CONCLUSION

and actuation signals in the $X$- and $Y$-axis, $R$ represents the sinusoidal chirp signal coming from the FFT analyzer. $-G$, $-\Sigma$, and $\Phi$ denote the inverting unity gain ($-1$), the inverting sum stage, and the first or second-order filter block, respectively. In this experiment, the clock rate is set to 2 MHz. A similar implementation scheme is used for the $Y$-axis. Note that the proposed scheme can be expanded to four FPAA's for simultaneous actuation of both axes.

Open-loop and closed-loop frequency responses of the MEMS nanopositioner with PLC and PPF controller are shown in Fig. 7. Comparing the open-loop and closed-loop frequency responses, we notice an improvement of approximately 30 dB in the resonant response of the device. We also observe that closed-loop responses with the damping controllers meet the set objectives of the control synthesis problem.

We also note that the cross-coupling effect at the first resonance frequency of the nanopositioner is alleviated, which reduces the vibration induced by the resonant mode. However, in Fig. 7(a), we observe a higher cross-coupling effect between the lateral axes at low frequencies with the PPF damping loop. This is due to the higher controller gain that is needed to achieve sufficient damping. Further improvement in tracking performance, if required, can be achieved by incorporating high-gain integral action in an outer feedback loop.

In Fig. 8, we compare open-loop and closed-loop responses of the system when a pulse-shaped voltage is applied to the actuators of the MEMS nanopositioner. In these experiments, controllers are implemented digitally in a DSPACE prototyping system. We notice a significant improvement in the transient response of the system with the damping controllers, protecting the device against unexpected disturbances. In particular, the closed-loop settling time is reduced to well below 1 ms with hardly any oscillations.
of the problem. Therefore, the polynomials $X$ and $Y$ are determined as

$$
X = \left( T^2\psi_1\psi_1^T \right) z^2 + \left( 2T^2\psi_1\psi_1^T \right) z + T^2\psi_1\psi_1^T
$$

$$
Y = \left( 4 + 2Td_1 + T^2\alpha_1^2 \right) I z^2 + \left( 2T^2\alpha_1^2 - 8 \right) z
$$

$$
\begin{cases}
X_2 = \frac{x_2}{y_2} \\
X_1 = \frac{x_1}{y_1} \\
Y = \frac{y_0}{y_1}
\end{cases}
$$

(40)

where $I$ is the identity matrix. We observe that $X_2 = X_0$ and $X_1 = 2X_0$, $X_2 > 0$. Also, we have $Y_0, Y_1 > 0$ assuming that $d_1$, $\alpha_1 > 0$.

**APPENDIX B**

Since $\Gamma(\omega) \in \mathbb{R}^{2 \times 2}$, we can rewrite (21) as

$$
[GY(Y - GX)^{-1} - T_d][GY(Y - GX)^{-1} - T_d]^* < \Gamma(\omega)
$$

$$
\Rightarrow [GY - T_d(Y - GX)][Y - GX]^{-1}[Y - GX]^{-*} \times [GY - T_d(Y - GX)]^* < \Gamma(\omega)
$$

$$
\Rightarrow [GY - T_d(Y - GX)][Y - GX]^* (Y - GX)^{-1} \times [GY - T_d(Y - GX)]^* < \Gamma(\omega)
$$

where $(\cdot)^{-*}$ represents the inverse conjugate transpose of a matrix. Assuming that $Z = Y - GX$, we can linearize $(Y - GX)^* (Y - GX)$ around an arbitrary matrix $Z_c \in \mathbb{C}^{2 \times 2}$ using the Taylor expansion [3], i.e.,

$$
Z^* Z \approx Z^* Z_c + Z_c^* Z - Z_c^* Z_c.
$$

(42)

Accordingly, we have

$$
[GY - T_d Z][Z^* Z_c + Z_c^* Z - Z_c^* Z_c]^{-1}[GY - T_d Z]^* < \Gamma(\omega)
$$

(43)

which can be formulated as the following LMI constraint:

$$
\begin{bmatrix}
\Gamma(\omega) & (GY - T_d Z) \\
(GY - T_d Z)^* & Z^* Z_c + Z_c^* Z - Z_c^* Z_c
\end{bmatrix} > 0 \ \forall \omega \in \Omega
$$

(44)

where $Z_c = Y_c - GX_c$ and $K_c = X_c Y_c^{-1}$ is the initial controller [3].

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